The Continuity Equation and Differential Mass and Energy Balances
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The differential total mass balance or continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\rho) = 0 \qquad (1)$$

For incompressible fluid in Cartesian, cylindrical and spherical coordinates, respectively, this becomes:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \qquad (2)$$

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \qquad (3)$$

$$\frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} = 0 \qquad (4)$$

The general differential component mass and energy balances are:

$$\frac{\partial C_A}{\partial t} + \mathbf{u}\cdot\nabla C_A = \nabla\cdot(D_{AB}\nabla C_A) + r_A \qquad (5)$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \frac{\nabla\cdot(k\nabla T)}{\rho C_p} + \frac{\Phi}{\rho C_p} + \frac{\dot{q}}{\rho C_p} \qquad (6)$$

where t is time, C_A is the concentration of the component of interest (A), $\boldsymbol{u}$ is the velocity vector, D_{AB} is the diffusivity of solute A in solvent B, r_A is the volumetric rate of generation of A by chemical reaction, T is temperature, k is the thermal conductivity, ρ is the density, C_p is the heat capacity, Φ is viscous heat generation and $\dot{q}$ is the volumetric rate of heat generation by absorption of radiation and chemical reaction.

For constant D_{AB} and k these equations reduce to the following in Cartesian coordinates:

$$\frac{\partial C_A}{\partial t} + u_x\frac{\partial C_A}{\partial x} + u_y\frac{\partial C_A}{\partial y} + u_z\frac{\partial C_A}{\partial z} = D_{AB}\left(\frac{\partial^2 C_A}{\partial x^2} + \frac{\partial^2 C_A}{\partial y^2} + \frac{\partial^2 C_A}{\partial z^2}\right) + r_A \qquad (7)$$

$$\frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\Phi}{\rho C_p} + \frac{\dot{q}}{\rho C_p} \qquad (8)$$

and in cylindrical coordinates:

$$\frac{\partial C_A}{\partial t} + u_r\frac{\partial C_A}{\partial r} + \frac{u_\theta}{r}\frac{\partial C_A}{\partial \theta} + u_z\frac{\partial C_A}{\partial z} = D_{AB}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 C_A}{\partial \theta^2} + \frac{\partial^2 C_A}{\partial z^2}\right) + r_A \qquad (9)$$

$$\frac{\partial T}{\partial t} + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + u_z\frac{\partial T}{\partial z} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\Phi}{\rho C_p} + \frac{\dot{q}}{\rho C_p} \qquad (10)$$

and in spherical coordinates:

$$\frac{\partial C_A}{\partial t} + u_r \frac{\partial C_A}{\partial r} + \frac{u_\theta}{r}\frac{\partial C_A}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial C_A}{\partial \phi} = D_{AB}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial C_A}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial C_A}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 C_A}{\partial \phi^2}\right) + r_A \qquad (11)$$

$$\frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} = \alpha\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right) + \frac{\Phi}{\rho C_p} + \frac{\dot{q}}{\rho C_p} \qquad (12)$$

where the quantity $k/\rho C_P$ is the thermal diffusivity and is given the symbol α.