Homework Set 7 – Key – Version 1.0
BE-382, Winter '08-'09, Dr. C. S. Tritt

Due 1/19

Note that both *u* and *v* are used to represent velocities in the following problems.

1. Show that the following 2-D flow field satisfies the continuity equation for steady flow of an impressible fluid:

$$u_r = \frac{1}{2} r^{-1/2} \cos\frac{\theta}{2}$$

$$u_\theta = -\frac{1}{2} r^{-1/2} \sin\frac{\theta}{2}$$

The cylindrical form of the continuity equation is:

$$\frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

The *z* term can be dropped because this is 2-D flow and the resulting equation multiplied by *r* to give.

$$\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0$$

The given functions are:

$$u_r = \frac{1}{2} r^{-1/2} \cos\frac{\theta}{2}$$

So,

$$\frac{\partial(r u_r)}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2} r^{1/2} \cos\frac{\theta}{2}\right) = \frac{1}{4} r^{-1/2} \cos\frac{\theta}{2}$$

And

$$u_\theta = -\frac{1}{2} r^{-1/2} \sin\frac{\theta}{2}$$

So,

$$\frac{\partial (u_\theta)}{\partial \theta} = -\frac{1}{4} r^{-\frac{1}{2}} \cos\frac{\theta}{2}$$

Inserting these back into the continuity equations gives the identity 0 = 0 so continuity is satisfied.

2. Write the equations and boundary conditions you would use to find the steady-state velocity distribution throughout a short, circular tube. Call the length of the tube *L*, its radius *R*. Assume a flat inlet velocity distribution ($v_z = v_0$ at all $r < R$, $z = 0$) and incompressibility.

Use cylindrical coordinates, so continuity is:

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \underbrace{\cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}}_{1} + \frac{\partial u_z}{\partial z} = 0$$

and the Navier-Stokes equations are:

r direction

$$\rho\left(\underbrace{\cancel{\frac{\partial v_r}{\partial t}}}_{3} + v_r\frac{\partial v_r}{\partial r} + \underbrace{\cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}}_{1} - \cancel{\frac{v_\theta^2}{r}} + v_z\frac{\partial v_r}{\partial z}\right) = \underbrace{\cancel{-\frac{\partial p}{\partial r}}}_{4} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}}_{1} - \underbrace{\cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}}_{1} + \frac{\partial^2 v_r}{\partial z^2}\right] + \underbrace{\cancel{\rho g_r}}_{2}$$

θ direction

$$\rho\left(\underbrace{\cancel{\frac{\partial v_\theta}{\partial t}}}_{3} + \cancel{v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}}\right) = \underbrace{\cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]}}_{1} + \underbrace{\cancel{\rho g_\theta}}_{2}$$

z direction

$$\rho\left(\underbrace{\cancel{\frac{\partial v_z}{\partial t}}}_{3} + v_r\frac{\partial v_z}{\partial r} + \underbrace{\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}}_{1} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}}_{1} + \frac{\partial^2 v_z}{\partial z^2}\right] + \underbrace{\cancel{\rho g_z}}_{2}$$

Terms are eliminated for the following reasons:

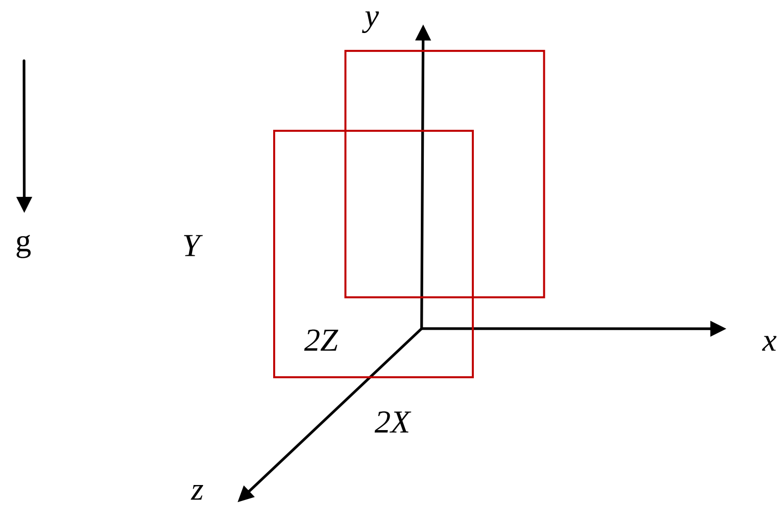
1. Circumferential uniformity (assumption)
2. Neglect gravity (assumption)
3. Steady state (given)
4. No radial pressure drop (assumption)

Steady flow problems do not require initial conditions. The boundary conditions are:

$v_z = v_0$ at all $r < R$, $z = 0$ (given)
$\frac{\partial v_z}{\partial z} = 0$ at z = 0 (assumed)
$v_z = 0$ at r = R, 0 < z < L (no slip)
$v_r = 0$ at r = R, 0 < z < L (no penetration)

$\frac{\partial v_z}{\partial r} = 0$ at r = 0 (symmetry)
$v_r = 0$ at r = 0 (symmetry)

3. Write the equations boundary and initial conditions you'd use to find the transient velocity distribution in a fluid flowing through a finite slit. Neglect pressure differences, but not gravity. Assume the fluid is initially at rest and at time zero the bottom of the slit is opened allowing the fluid to move (prior to this, there would be a hydrostatic pressure gradient that would counter gravity). Assume a uniform inlet velocity distribution. Assume the slit is oriented as shown below.



Note the drawing is poor because Word's drawing tool really sucks. The vertices of the two red rectangles should be connected. The slit is a $2X$ by Y by $2Z$ rectangular prism. The faces in the y-x and y-z planes are solid. The openings are in the x-z plane at $y = 0$ and $y = Y$.

Based on geometry, decide to use Cartesian form of equations. Based on symmetry only solve problem for the region $0 < x < X$, $0 < z < Z$ and $0 < y < Y$. Once the solution is known in this region, it can be generalized to the other ¾'s of the slit.

Continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

N-S Equations

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\cancel{\frac{\partial p}{\partial x}}^{\,2} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \cancel{\rho g_x}^{\,1}$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\cancel{\frac{\partial p}{\partial y}}^{\,2} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\cancel{\frac{\partial p}{\partial z}}^{\,2} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \cancel{\rho g_z}^{\,1}$$

Eliminate terms based on the following:

1. Gravity acts only in y direction.
2. Neglect pressure gradients (given).

Boundary Conditions

$v_x = v_y = v_z = 0$ at $x = X$, $0 < y < Y$, $0 < z < Z$ (y-z face, no slip and no penetration)

$v_x = v_y = v_z = 0$ at $0 < x < X$, $0 < y < Y$, $z = Z$ (x-y face, no slip and no penetration)

$\frac{\partial v_y}{\partial z} = \frac{\partial v_z}{\partial z} = 0$ at z = 0, $0 < x < X$, $0 < y < Y$ (symmetry)

$\frac{\partial v_y}{\partial x} = \frac{\partial v_x}{\partial x} = 0$ at $x = 0$, $0 < y < Y$, $0 < z < Z$ (symmetry)

$v_y = v_0$ and $v_x = v_z = 0$ at $y = Y$, $0 < x < X$, $0 < z < Z$ (given, but may not be valid)

Initial Conditions

$v_x = v_y \;\; v_z = 0$ for $0 < x < X$, $0 < z < Z$ and $0 < y < Y$.